

Simple state preparation for contextuality tests with few observables

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We consider any noncontextuality inequality, and the state preparation scheme which consists in performing any von Neumann measurement on any initial state. For an inequality which is not always satisfied, and Hilbert space dimensions greater than a value specified by the inequality, we determine necessary and sufficient conditions for the existence of observables with which the inequality is violated after the preparation process. For an initial state with no zero eigenvalues, there are always such observables, and which are independent of this state.

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Quantum mechanics is contextual. Measurement outcomes are not predetermined independently of the measurements actually performed [1, 2]. This can be revealed with a finite set of observables, such that each observable is compatible with some other ones, but not with all. When evaluated with a noncontextual hidden-variable theory, the correlations of the compatible observables, satisfy inequalities, which can be violated by quantum systems. Well-known examples of such noncontextuality inequalities are Clauser-Horne-Shimony-Holt (CHSH) and Klyachko-Can-Binicioğlu-Shumovsky (KCBS) inequalities [3–6]. Inequalities have been obtained, which are disobeyed for any state of the considered system. Moreover, this holds for fixed sets of observables, independent of the state. These inequalities involve 13 dichotomic observables for a three-level system [7–9], and 9 for a four-level system [10–12]. However, contextuality can be revealed with less observables, 4 are enough for a d -level system with $d \geq 4$, using CHSH inequality.

But contextuality tests with few observables, have two drawbacks. First, for some states, the noncontextuality inequality is satisfied with any set of observables obeying the required compatibility relations. Second, for the other states, the observables must be chosen according to the state, in order to violate the inequality [13–17]. A noncontextuality inequality involves a sum of expectation values. Thus, if it cannot be violated for pure states, it is always satisfied, and is hence not a proper contextuality test. The maximally mixed state of the considered system also plays a particular role. For three-level systems, an inequality, with 9 observables, has been found, that can be violated for any state except the maximally mixed one [15]. The relation between the mixedness of a state, in the sense of majorization [18, 19], and its usefulness for revealing quantum contextuality, has been clarified in Ref.[16]. The eigenvalues of the state do not alone dictate if a given noncontextuality inequality can be disobeyed. The dimension of the system Hilbert space is an important parameter, since it determines the set of potentially accessible observables [20].

In this Letter, we show that the two above mentioned

drawbacks do not mean that violating a given noncontextuality inequality necessarily requires a very efficient state preparation. We consider the state preparation scheme which consists in starting from any initial state, and performing any von Neumann measurement. When the dimension d of the system Hilbert space is greater than a value, determined by the inequality, which is, e.g., 4 for CHSH inequality, and the inequality is a proper contextuality test, it cannot be violated after the preparation measurement, if only if a single outcome of this measurement has nonzero probability, and the inequality is always satisfied for the initial state. This exceptional case is easy to detect, and cannot occur if the ranks of the measurement projectors are lower than d minus a constant, specified by the inequality. This constant is equal to 2 for CHSH inequality, for example. Moreover, when the initial state has no zero eigenvalues, the inequality is necessarily violated after the preparation process, and with observables independent of this state.

A noncontextuality inequality, with N dichotomic observables A_k , reads

$$\sum_n x_n \left\langle \prod_{k \in \mathcal{E}_n} A_k \right\rangle \leq 1, \quad (1)$$

where $\langle \dots \rangle = \text{Tr}(\rho \dots)$ denotes the average with respect to the quantum state ρ . The subsets $\mathcal{E}_n \subset \{1, \dots, N\}$, are such that $[A_k, A_l] = 0$ when k and l both belong to \mathcal{E}_n . The number of terms in the sum and the coefficients x_n depend on the inequality considered. A noncontextuality inequality is always satisfied when the observables A_k are replaced by classical random variables $a_k = \pm 1$, and the average is evaluated with respect to a probability distribution of these variables. Moreover, the coefficients x_n are such that the maximum value of the left-hand side of eq.(1) with classical random variables, is 1. Thus, a violation of inequality (1) clearly indicates that the obtained value cannot be accounted for by a noncontextual hidden-variable theory.

The states ρ of a d -level system, for which a given inequality (1) is violated with appropriate observables

A_k , are determined by the function C_d defined as

$$C_d(\rho) = \max_{\mathbf{A} \in \mathcal{A}_d} \text{Tr}[\rho T(\mathbf{A})], \quad (2)$$

where $\mathbf{A} = (A_1, \dots, A_N)$, $T(\mathbf{A}) = \sum_n x_n \prod_{k \in \mathcal{E}_n} A_k$, and \mathcal{A}_d denotes the set of all \mathbf{A} consisting of dichotomic observables A_k , of the d -level system, which obey $[A_k, A_l] = 0$ for $k, l \in \mathcal{E}_n$. Note that this definition depends on the dimension d of the considered Hilbert space. By construction, for a state ρ such that $C_d(\rho) \leq 1$, inequality (1) is satisfied with any dichotomic observables A_k obeying the required commutation relations.

It results directly from the definition (2) that the function C_d is continuous, invariant under unitary transformations of ρ , and convex.

Proposition 1. *For any states ρ_m , unitary operator U , and probabilities p_m such that $\sum_m p_m = 1$,*

$$i) |C_d(\rho_1) - C_d(\rho_2)| \leq \sqrt{d} \sum_n |x_n| \text{Tr}[(\rho_1 - \rho_2)^2]^{1/2},$$

$$ii) C_d(U\rho_1 U^\dagger) = C_d(\rho_1),$$

$$iii) C_d(\sum_m p_m \rho_m) \leq \sum_m p_m C_d(\rho_m).$$

Proof. i) It follows from the Cauchy-Schwarz inequality that $\text{Tr}[\omega T(\mathbf{A})]^2 \leq \text{Tr}(\omega^2) \sum_{i=1}^d t_i^2$, where $\omega = \rho_1 - \rho_2$, and t_i denotes the eigenvalues of $T(\mathbf{A})$. Since $|\langle \prod_k A_k \rangle| \leq 1$ for any state ρ , $|t_i| \leq \sum_n |x_n|$. Consequently, for any $\mathbf{A} \in \mathcal{A}_d$, $\text{Tr}[\rho_{1/2} T(\mathbf{A})] \leq C_d(\rho_{2/1}) + [d \text{Tr}(\omega^2)]^{1/2} \sum_n |x_n|$. Maximizing over \mathbf{A} completes the proof of point i).

ii) $\text{Tr}[U\rho_1 U^\dagger T(\mathbf{A})] = \text{Tr}[\rho_1 T(\mathbf{B})]$ where $B_k = U^\dagger A_k U$. For $\mathbf{A} \in \mathcal{A}_d$, the observables B_k satisfy the commutation relations $[B_k, B_l] = U^\dagger [A_k, A_l] U = 0$ for $k, l \in \mathcal{E}_n$, and are dichotomic, since $B_k^2 = U^\dagger A_k^2 U = I_d$ where I_d is the d -dimensional identity operator. Consequently, \mathbf{B} belongs to \mathcal{A}_d . Thus, the above equality yields $\text{Tr}[U\rho_1 U^\dagger T(\mathbf{A})] \leq C_d(\rho_1)$ for any $\mathbf{A} \in \mathcal{A}_d$, and hence $C_d(U\rho_1 U^\dagger) \leq C_d(\rho_1)$. Since this inequality is valid for any ρ_1 and U , the equality holds for any ρ_1 and U .

iii) By linearity of the trace, $\text{Tr}[\sum_m p_m \rho_m T(\mathbf{A})] \leq \sum_m p_m C_d(\rho_m)$ for any $\mathbf{A} \in \mathcal{A}_d$, which proves iii). \square

From point i) of proposition 1, it ensues that, any state ρ such that $C_d(\rho) > 1$, has a neighborhood of states which can violate eq.(1). Thus, no noncontextuality inequality can be disobeyed only for pure states. Points ii) and iii) show that applying unitary transformations to states ρ such that $C_d(\rho) \leq 1$, or preparing statistical mixtures of such states, cannot lead to a violation of inequality (1). Another result of point ii) is that $C_d(\rho)$ depends only on the eigenvalues of the state ρ .

The convexity and invariance under unitary transformations of C_d have the following consequence for positive operator-valued measurements. From now on, we use the notation

$$C_d(F, \rho) = C_d[F\rho F^\dagger / \text{Tr}(F^\dagger F \rho)], \quad (3)$$

where ρ is a state, and F any operator such that $F\rho F^\dagger \neq 0$, of a d -level system.

Corollary 1. *For any state ρ , and operators F_m such that $\sum_m F_m^\dagger F_m = I_d$, of a d -level system, $C_d(\rho) \leq \max_{m \in \mathcal{E}} C_d(F_m, \rho)$, where $\mathcal{E} = \{m : F_m \rho F_m^\dagger \neq 0\}$.*

Proof. There are unitary operators U_m such that $\rho = \sum_{m \in \mathcal{E}} p_m U_m \rho_m U_m^\dagger$ where $p_m = \text{Tr}(F_m^\dagger F_m \rho)$ and $\rho_m = F_m \rho F_m^\dagger / p_m$ [21]. Thus, with proposition 1, $C_d(\rho) \leq \sum_{m \in \mathcal{E}} p_m C_d(\rho_m)$, which leads, with $\sum_{m \in \mathcal{E}} p_m = 1$, to the result. \square

In other words, for any measurement, at least one resulting state ρ_m gives a value of C_d which can exceed that of the initial state. However, this obviously does not guarantee that inequality (1) can be violated. In the following, we show conditions under which this is the case for von Neumann measurements, i.e., if the operators F_m are projectors.

Below, we make use of the majorization relation, which is defined as follows. Consider two real d -component vectors \mathbf{a} and \mathbf{b} , and the vectors \mathbf{a}^\downarrow and \mathbf{b}^\downarrow obtained from \mathbf{a} and \mathbf{b} , respectively, by rearranging their components in decreasing order, i.e., $a_i^\downarrow \geq a_{i+1}^\downarrow$. It is said that \mathbf{a} majorizes \mathbf{b} , denoted $\mathbf{a} \succ \mathbf{b}$, iff, for $j = 1, \dots, d$, $\sum_{i=1}^j a_i^\downarrow \geq \sum_{i=1}^j b_i^\downarrow$, with equality for $j = d$. For density matrices, $\rho_1 \succ \rho_2$ iff $\lambda(\rho_1) \succ \lambda(\rho_2)$, where the spectrum $\lambda(A)$ is the vector made up of the eigenvalues of the Hermitian operator A , in decreasing order [18, 19]. The majorization relation is generalized to states of systems of different sizes, by extending with zeros the spectrum with less eigenvalues. The next proposition will be proved using the following lemma.

Lemma. *Consider three real d -component vectors \mathbf{a} , \mathbf{b} and \mathbf{c} . If $\mathbf{a} \succ \mathbf{b}$ then $\mathbf{b} \cdot \mathbf{c} \leq \mathbf{a}^\downarrow \cdot \mathbf{c}^\downarrow$, where $\mathbf{a} \cdot \mathbf{b} = \sum_{i=1}^d a_i b_i$.*

Proof. It is already known that $\mathbf{b} \cdot \mathbf{c} \leq \mathbf{b}^\downarrow \cdot \mathbf{c}^\downarrow$ [18, 19]. We define $R_j = \sum_{i=1}^j (b_i^\downarrow - a_i^\downarrow)$ for $j = 1, \dots, d$. Since $\mathbf{a} \succ \mathbf{b}$, $R_j \leq 0$ and $R_d = 0$. Thus, $(\mathbf{b}^\downarrow - \mathbf{a}^\downarrow) \cdot \mathbf{c}^\downarrow = \sum_{j=1}^{d-1} (c_j^\downarrow - c_{j+1}^\downarrow) R_j \leq 0$. \square

To investigate the influence of the Hilbert space dimension d , we define the application $\mathcal{G} : \mathbf{A}' \mapsto \mathbf{A}$ as follows. As mentioned above, for any noncontextuality inequality (1), there is (a_1, \dots, a_N) such that $a_k = \pm 1$ and $\sum_n x_n \prod_{k \in \mathcal{E}_n} a_k = 1$. For any $\mathbf{A}' \in \mathcal{A}_{d'}$, $\mathbf{A} = (A_1, \dots, A_N) \in \mathcal{A}_d$, is given by $\langle i | A_k | j \rangle = \langle i | A'_k | j \rangle$, for $d = d'$, and by

$$A_k = \sum_{i,j=1}^{d'} \langle i | A'_k | j \rangle |\tilde{i}\rangle \langle \tilde{j}| + a_k \sum_{i=d'+1}^d |\tilde{i}\rangle \langle \tilde{i}|, \quad (4)$$

for $d > d'$, where $\{|\tilde{i}\rangle\}_{i=1}^{d'}$ and $\{|\tilde{i}\rangle\}_{i=1}^d$ are orthonormal bases of the considered Hilbert spaces. With matrix representations of the observables A_k and A'_k , it is straightforward to show that, when $\mathbf{A}' \in \mathcal{A}_{d'}$, the observables

A_k are dichotomic and obey the required commutation relations, and that the spectrum of $T(\mathbf{A})$ consists of the d' eigenvalues $\lambda_i[T(\mathbf{A}')]]$ and of $d - d'$ ones. Using the lemma and function \mathcal{G} , the following can be shown.

Proposition 2. *Consider a state ρ of a d -level system, and a state ρ' of a d' -level system. If $\rho \succ \rho'$ and $d \geq d'$ then $C_d(\rho) \geq C_{d'}(\rho')$.*

Proof. We first prove that

$$C_d(\rho) = \max_{\mathbf{t} \in \Lambda_d} [\mathbf{t} \cdot \boldsymbol{\lambda}(\rho)] \quad (5)$$

where $\Lambda_d = \{\boldsymbol{\lambda}[T(\mathbf{A})] : \mathbf{A} \in \mathcal{A}_d\}$. For that purpose, we write $\text{Tr}[\rho T(\mathbf{A})] = \sum_{i=1}^d t_i p_i$ where $p_i = \langle i|\rho|i\rangle$, $|i\rangle$ denotes the eigenvectors of $T(\mathbf{A})$, and \mathbf{t} its spectrum. The Schur-Horn theorem gives $\boldsymbol{\lambda}(\rho) \succ (p_1, \dots, p_d)$ [19]. Thus, using the lemma, we obtain $\text{Tr}[\rho T(\mathbf{A})] \leq \mathbf{t} \cdot \boldsymbol{\lambda}(\rho)$, which results in $C_d(\rho) \leq \max_{\mathbf{t} \in \Lambda_d} [\mathbf{t} \cdot \boldsymbol{\lambda}(\rho)]$.

Consider $\mathbf{t} \in \Lambda_d$. By definition of Λ_d , there is $\mathbf{A} \in \mathcal{A}_d$ such that $\boldsymbol{\lambda}[T(\mathbf{A})] = \mathbf{t}$. Consider \mathbf{B} defined by $B_k = U^\dagger A_k U$ where U is any unitary operator. \mathbf{B} belongs to \mathcal{A}_d (see proof of point ii) of proposition 1). Moreover, $T(\mathbf{B}) = \sum_{i=1}^d t_i U^\dagger |i\rangle \langle i| U$. Therefore, there is $\tilde{\mathbf{A}} \in \mathcal{A}_d$ such that the spectrum of $T(\tilde{\mathbf{A}})$ is \mathbf{t} , and its eigenvectors are identical to those of ρ . Hence, $\sum_{i=1}^d t_i \lambda_i(\rho) = \text{Tr}[\rho T(\tilde{\mathbf{A}})] \leq C_d(\rho)$, which gives the second inequality required to prove eq.(5).

Consider a state $\tilde{\rho}'$ of a d -level system, with the same nonzero eigenvalues than ρ' . Since $\rho \succ \rho'$, $\boldsymbol{\lambda}(\rho) \succ \boldsymbol{\lambda}(\tilde{\rho}')$. Thus, using the lemma and the form (5), we have $\mathbf{t} \cdot \boldsymbol{\lambda}(\tilde{\rho}') \leq C_d(\rho)$ for any $\mathbf{t} \in \Lambda_d$.

For any $\mathbf{A}' \in \mathcal{A}_{d'}$, expression (4) gives $\mathbf{A} \in \mathcal{A}_d$ such that the components of $\mathbf{t} = \boldsymbol{\lambda}[T(\mathbf{A})]$ are the d' eigenvalues $t'_i = \lambda_i[T(\mathbf{A}')]]$, and $d - d'$ ones, arranged in decreasing order. Thus, $t_i = t'_i$ if $t'_i \geq 1$, and $t_i = 1$ or t'_{i-j} where $j \geq d - d'$, if $t'_i < 1$. So, for $i \leq d'$, $t'_i \leq t_i$, and hence $\text{Tr}[\rho' T(\mathbf{A}')] \leq \sum_{i=1}^{d'} t'_i \lambda_i(\rho') \leq \sum_{i=1}^{d'} t_i \lambda_i(\tilde{\rho}')$, which, together with the above inequality, leads to the result. \square

For a von Neumann measurement, the resulting states $\rho_m = \Pi_m \rho \Pi_m / \text{Tr}(\Pi_m \rho)$ where Π_m are projectors and ρ is the initial state, have vanishing eigenvalues. To study their ability to violate inequality (1), it is convenient to define

$$C_d^{(r)} = \max_{\mathbf{t} \in \Lambda_d} \sum_{i=1}^r t_i / r, \quad (6)$$

where $r \leq d$. Noting that $C_d^{(r)} = C_d(\Pi/r)$ where Π is any rank- r projector, see eq.(5), and using proposition 2, the following properties of $C_d^{(r)}$ and C_d can be proved. Any density matrix ρ of rank r , satisfies $\rho \succ \Pi/r$, and hence $C_d(\rho) \geq C_d^{(r)}$. Consequently, if $C_d^{(r)} > 1$, a d -level system in such a state ρ , can violate inequality (1). Since $\Pi/r \succ \Pi'/r'$ where $r' \geq r$ and Π' is a rank- r' projector, $C_d^{(r)}$ decreases as r increases, and increases with d .

The function C_d reaches its maximum $C_d^{(1)}$ for pure states, which majorize any other state, and its minimum $C_d^{(d)}$ for the maximally mixed state I_d/d , which is majorized by any state of rank not larger than d . Thus, these two extreme values determine, for a d -level system, whether inequality (1) can be disobeyed or not, for all states or not. If $C_d^{(1)} \leq 1$, eq.(1) is satisfied with any observables A_k obeying the required commutation relations, for any system state ρ . It is then not a proper contextuality test for dimension d . If $C_d^{(d)} > 1$, inequality (1) can be violated for any state ρ , but it may remain necessary to choose the observables A_k according to ρ . If $C_d^{(d)} \leq 1 < C_d^{(1)}$, observables A_k can be found to disobey eq.(1) or not, depending on the spectrum of ρ .

Since $C_d^{(1)}$ increases with d , if inequality (1) is a contextuality test for a dimension d' , it is also so for dimensions $d \geq d'$. Relations (4) lead to $C_d^{(d)} \geq 1 + (C_{d'}^{(d')} - 1)d'/d$, for $d \geq d'$. Thus, if eq.(1) can be violated for any state of a d' -level system, this is also the case for a larger system. The increase with d of $C_d^{(r)}$ obviously does not guarantee that it exceeds 1 for large enough d . Below, we show that this is actually the case, under the only assumption that inequality (1) is not always satisfied, and draw consequences for von Neumann preparation measurements.

Proposition 3. *Consider a state ρ , and projectors Π_m such that $\sum_m \Pi_m = I_d$, their ranks are not larger than r , and the rank of Π_1 is r , of a d -level system, and define \mathcal{E} the set of m such that $\text{Tr}(\Pi_m \rho) \neq 0$. Assume there is d' such that $C_{d'}^{(1)} > 1$, and $d \geq d'$.*

- i) *If $r \leq d - d' + 1$, then $C_d(\Pi_m, \rho) > 1$ for any $m \in \mathcal{E}$.*
- ii) *If $r > d - d' + 1$, $d \geq 2d' - 3$, and $\text{Tr}(\Pi_1 \rho) \neq 1$, then $C_d(\Pi_m, \rho) > 1$ for at least one $m \in \mathcal{E}$.*

Proof. We first show that $C_d^{(r)} > 1$ for $r \leq d - d' + 1$. Since $C_{d'}^{(1)} > 1$, there is $\mathbf{A}' \in \mathcal{A}_{d'}$ such that $t'_1 > 1$, where $\mathbf{t}' = \boldsymbol{\lambda}[T(\mathbf{A}')]]$, see eq.(6). Consider $\mathbf{A} \in \mathcal{A}_d$, following from eq.(4), and denote by \mathbf{t} the spectrum of $T(\mathbf{A})$. We have $t_1 = t'_1$ and $t_i \geq 1$ for $i \leq d - d' + 1$. Consequently, $\sum_{i=1}^r t_i / r \geq 1 + (t'_1 - 1)/r > 1$.

We define $\rho_m = \Pi_m \rho \Pi_m / \text{Tr}(\Pi_m \rho)$ for any $m \in \mathcal{E}$.

i) Since $\rho_m \succ \Pi_m / r_m$ where r_m is the rank of Π_m , and $r_m \leq r$, proposition 2 gives $C_d(\rho_m) \geq C_d^{(r_m)} \geq C_d^{(r)}$. So, using the above result, we get $C_d(\rho_m) > 1$.

ii) There is $m \neq 1$ such that $\text{Tr}(\Pi_m \rho) \neq 0$. The rank r_m of ρ_m is not larger than $d - r \leq d - d' + 1$, and hence, $C_d(\rho_m) \geq C_d^{(r_m)} > 1$. \square

For dimensions $d \geq 2d' - 3$ where d' is such that $C_{d'}^{(1)} > 1$, it results from proposition 3 that inequality (1) cannot be violated after the preparation measurement, only if a single outcome of this measurement has nonzero probability, and $C_d(\rho) \leq 1$ where ρ is the initial

state. This last condition comes from the fact that the sole post-measurement state is equal to ρ .

Corollary 2. *Consider a state ρ , and projectors Π_m such that $\sum_m \Pi_m = I_d$, of a d -level system. Assume there is d' such that $C_{d'}^{(1)} > 1$, and $d \geq 2d' - 3$.*

$C_d(\Pi_m, \rho) \leq 1$ for all Π_m such that $\text{Tr}(\Pi_m \rho) \neq 0$ iff $\text{Tr}(\Pi_m \rho) = 1$ for one m , and $C_d(\rho) \leq 1$.

Proof. If $\text{Tr}(\Pi_m \rho) = 1$, then $\text{Tr}(\Pi_{m'} \rho) = 0$ for any $m' \neq m$, and $\rho = \sum_{m', m''} \Pi_{m'} \rho \Pi_{m''} = \Pi_m \rho \Pi_m$.

If $C_d(\Pi_{m'}, \rho) \leq 1$ for all appropriate $\Pi_{m'}$, then, due to proposition 3, $\text{Tr}(\Pi_m \rho) = 1$ where Π_m is the projector of largest rank, and thus $\rho = \Pi_m \rho \Pi_m$. \square

Proposition 3 concerns the post-measurement states ρ_m , and the possibility to violate inequality (1) after one of them was selected. If, on the contrary, the measurement is unread, the state of the system after it, is $\rho' = \sum_m \Pi_m \rho \Pi_m$, which obeys $\rho \succ \rho'$, due to quantum Hardy-Littlewood-Pólya theorem [21]. It follows from proposition 2, that eq.(1) is always obeyed for ρ' if $C_d(\rho) \leq 1$. Interesting measurements are dichotomic ones with projectors of ranks $d/2$ for even d , and $(d \pm 1)/2$ for odd d . They are the most inefficient in the sense that there is a projector Π_m of lower rank for all the other measurements. For these measurements, inequality (1) can be disobeyed for both resulting states, provided $d \geq 2d' - 2$. The smaller is the dimension d' , the less demanding are the conditions in proposition 3. The minimum d' such that $C_{d'}^{(1)} > 1$ is 4 for CHSH inequality [4], and 3 for KCBS inequality [6].

Proposition 3 ensures that, for any state ρ with no zero eigenvalues, and any von Neumann measurement, there are observables A_k such that inequality (1) is violated for a post-measurement state ρ_m , provided $d \geq 2d' - 3$. Such observables A_k depend a priori on ρ . We show below that some of them are determined only by the considered measurement and inequality.

Proposition 4. *Consider projectors Π_m such that $\sum_m \Pi_m = I_d$, of a d -level system.*

If there is d' such that $C_{d'}^{(1)} > 1$, and $d \geq 2d' - 3$, then there are $\mathbf{A} \in \mathcal{A}_d$ and a projector Π_m , such that $\text{Tr}[\rho_m T(\mathbf{A})] > 1$ where $\rho_m = \Pi_m \rho \Pi_m / \text{Tr}(\Pi_m \rho)$, for all states ρ with no zero eigenvalues, of the d -level system.

Proof. There is at least one projector Π_m of rank $r \leq d/2$. Denote by $|\phi\rangle$ one of its eigenvectors with eigenvalue 1. Since $C_{d'}^{(1)} > 1$, there is $\mathbf{B}' \in \mathcal{A}_{d'}$ such that $t'_1 > 1$, where $\mathbf{t}' = \lambda[T(\mathbf{B}')]$. Consider $\mathbf{B} \in \mathcal{A}_d$, following from eq.(4), and define $\mathbf{t} = \lambda[T(\mathbf{B})]$. We have $t_1 = t'_1$, and, since $r \leq d - d' + 1$, $t_i \geq 1$ for $i \leq r$. There is $\mathbf{A} \in \mathcal{A}_d$ such that the spectrum of $T(\mathbf{A})$ is \mathbf{t} , its first r eigenvectors $|i\rangle$ obey $\Pi_m |i\rangle = |i\rangle$, and $|1\rangle = |\phi\rangle$ (see proof of proposition 2). $\text{Tr}(\Pi_m \rho) \geq r\lambda$, and $\langle \phi | \rho | \phi \rangle \geq \lambda$, where $\lambda = \min_j \lambda_j(\rho) > 0$. Finally, the above results lead to $\text{Tr}[\rho_m T(\mathbf{A})] \geq 1 + (t'_1 - 1)p$ where $p = \langle \phi | \rho | \phi \rangle / \text{Tr}(\Pi_m \rho) > 0$. \square

For an initial state $\rho = \sum_{i=1}^d \lambda_i(\rho) |i\rangle\langle i|$ of a D -level system, of rank $d < D$, proposition 4 holds for the subspace spanned by $\{|i\rangle\}_{i=1}^d$, if d is large enough. Thus, a violation of inequality (1) can be achieved knowing only the subspace corresponding to the zero, or very small, eigenvalues of ρ , and choosing the preparation measurement, and observables A_k , accordingly.

In summary, we have studied the possibility of preparing a state that violates a given noncontextuality inequality, by performing any von Neumann measurement on any initial state. For a large enough system, and an inequality which is not always satisfied, we have determined necessary and sufficient conditions for the existence of observables with which the inequality is violated for a state resulting from the preparation measurement. For an initial state with no zero eigenvalues, there are always such observables, and which do not depend on this state. A natural extension of this work is to consider, for the preparation stage, general measurements, for which only a partial result has been obtained.

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